ON AN OPTIMAL CONTROL PROBLEM

(OB ODNOI ZADACHE OPTIMAL'NOGO REGULIROVANIIA)

PMM Vol.26, No.1, 1962, pp. 181-184

L.S. GNOENSKII
(Moscow)

(Received July 7, 1961)

1. The following problem is encountered in the design of control systems. A system is described by the equations

$$\dot{x}_j + \sum_{k=1}^{n} a_{jk}(t) x_k = b_i u(t) \qquad (j = 1, \ldots, n) \tag{1}$$

$$|u(t)| \leqslant m \tag{2}$$

The solution $x\ (x_1, x_2, \ldots, x_n)$ satisfies the initial conditions $x = x_0\ (x_{10}, \ldots, x_{n0})$ at $t = 0$. A set N_k is given such that if $x\ (x_1, \ldots, x_n) \in N_k$, then

$$x_1 = a_1, \qquad x_2 = a_2, \ldots, \qquad x_k = a_k$$

Here $a_1, \ldots, a_k$ are constants. We shall assume that there exists a set V of control functions $u(t)$ satisfying (2) such that $u(t) \in V$, then the solution of the system (1) passes from the point x_0 into the set N_k. It is required to find in the set V a function $u_{min}(t)$ which takes the solution from the point x_0 into the set N_k in the shortest time. This problem has been considered in [1-5] and other articles, and the most general results have been obtained for it.

In this article we shall discuss a somewhat different method for finding $u_{min}(t)$ for the case $k = 2$, which, while a special case, is fairly important for applications. The method is based on a generalization of the method of accumulated perturbations.

The proposed method closely resembles a method mentioned in [3]. It has, however, certain particular features which appear likely to be useful in some applications.

As is known, a solution of the system (1) may be represented in the form

$$x_j(t) = x_{j0}(t) + \int_0^t K_j(t, \tau) u(\tau) d\tau, \qquad (j = 1, \ldots, n), \qquad x_{j0}(0) = x_{j0} \tag{3}$$

Let A_i be a set of points t belonging to the positive half-axis such that on A_i we have

$$m\int_0^t |K_i(t,\tau)|\,d\tau \geqslant |a_i - x_{i0}(t)| \equiv |n_i(t)| \qquad (i = 1, 2)$$

We shall denote by A the intersection of the sets A_1 and A_2. If $A = 0$, then we know that a transition from the point x_0 into the set N_2 is not feasible. Let us assume that $A \neq 0$, T is an arbitrary point of A, and $u_T(\tau)$ belongs to the set M of functions $u(\tau)$ satisfying (2) and the relation

$$\int_0^T K_1(T,\tau)\,u(\tau)\,d\tau = n_1(T) \tag{4}$$

For $u_T(\tau)$ we have

$$\operatorname{Inf}\left| n_2(T) - \int_0^T K_2(T,\tau)\,u(\tau)\,d\tau \right| \qquad (u \in M) \tag{5}$$

If the equation for T

$$n_2(T) - \int_0^T K_2(T,\tau)\,u_T(\tau)\,d\tau = 0 \tag{6}$$

has a solution, then a transition from the point x_0 into N_2 is feasible and the optimal transition with respect to speed is carried out by the function

$$u_{\min}(\tau) = u_{T_0}(\tau)$$

Here T_0 is the smallest root of Equation (6).

In Formulas (4) to (6) we may interchange the indices 1 and 2, but this will not change the value of T_0. It should be noted that it is sometimes sufficient to have the coordinate x_2 lie not at a_2 but in some ϵ-neighborhood of a_2. In the present method the value of (5) is determined. As soon as it is found for some $T < T_0$ that (5) is less than ϵ, we obtain a function $u_T(\tau)$ which carries out the transition from the point x_0 to the point whose first coordinate is equal to a_1 and whose second coordinate lies in an ϵ-neighborhood of a_2.

2. Let us construct the function $u_T(\tau)$. We set

$$K_1(T,\tau) = K_1(\tau), \qquad K_2(T,\tau) = K_2(\tau), \qquad K(\tau) = \frac{K_2(\tau)}{K_1(\tau)}$$

$$A^+ = \operatorname*{Sup}_\tau K(\tau), \qquad A^- = \operatorname*{Inf}_\tau K(\tau), \qquad \tau \in [0, T]$$

$$I_{ij}(x, y) = m \int_{\sigma(x,y)} K_i(\tau)\,\operatorname{sign} K_j(\tau)\,d\tau$$

where $\sigma(x, y)$ is a set belonging to $[0, T]$ such that if, $r \in \sigma(x, y)$, then $y > K(r) \geqslant x$. It is assumed that none of the functions $K_1(r)$, $K_2(r)$, $K(r)$ can be constant on sets of nonzero measure. These restrictions do not cause any difficulty.

The A^- and A^+ and A^- may be infinite. Since $T \in A$, we have

$$I_{11}(A^-, A^+) = m\int_0^T K_1(\tau)\operatorname{sign} K_1(\tau)\,d\tau = n_1(T) + \alpha_{11}(T), \quad \alpha_{11}(T) > 0$$

We shall denote by $u^+(r)$ and $u^-(r)$, respectively, the functions for which

$$n^+ = \sup_u \int_0^T K_2(\tau)\,u(\tau)\,d\tau, \qquad n^- = \inf \int_0^T K_2(\tau)\,u(\tau)\,d\tau, \qquad u \in M$$

The value $n_2(T)$ belongs to one of the intervals

$$(-\infty, n^-),\ [n^-, n^+],\ (n^+, \infty)$$

Obviously, if $n_2(T) \in (-\infty, n^-)$, then the function $u_T(r)$ for which we have (5) coincides with $u^-(r)$; if $n_2(T) \in (n^+, \infty)$, then $u_T(r)$ coincides with $u^+(r)$. It will be shown below that if $n_2(T) \in [n^-, n^+]$, then there exists a function $u_a(r)$ in the set M satisfying the equation

$$\int_0^T K_2(\tau)\,u_a(\tau)\,d\tau = n_2(T)$$

that is, $u_a(r)$ coincides with $u_T(r)$.

We shall prove that

$$u^+(\tau) = m\operatorname{sign} K_1(\tau) \quad \text{if } \tau \in \sigma(y_0, A^+)$$
$$u^+(\tau) = -m\operatorname{sign} K_1(\tau) \quad \text{if } \tau \in \sigma(A^-, y_0)$$

where y_0 satisfies the equation

$$I_{11}(A^-, y) = \frac{\alpha_{11}}{2} \tag{7}$$

$$u^-(\tau) = -m\operatorname{sign} K_1(\tau) \quad \text{if } \tau \in \sigma(y_1, A^+)$$
$$u^-(\tau) = m\operatorname{sign} K_1(\tau) \quad \text{if } \tau \in \sigma(A^-, y_1)$$

where y_1 satisfies the equation

$$I_{11}(y, A^+) = \frac{\alpha_{11}}{2} \tag{8}$$

It should be noted that for any y in $[A^-, A^+]$ the relation

$$\sigma(A^-, y) \cup \sigma(y, A^+) = [0, T]$$

is satisfied.

Taking into consideration the above mentioned properties of the functions $K_1(r)$ and $K(r)$, we find that $I_{11}(A^-, y)$ and $I_{11}(y, A^+)$ are continuous and strictly monotonic functions of y. Since

$$I_{11}(A^-, A^-) = 0, \qquad I_{11}(A^+, A^+) = 0, \qquad I_{11}(A^-, A^+) = n_1(T) + \alpha_{11}(T)$$

each of the Equations (7) and (8) has a unique solution in (A^-, A^+).

The functions $u^+(r)$ and $u^-(r)$ belong to M.

In fact:

$$\int_0^T K_1(\tau)\, u^+(\tau)\, d\tau = -I_{11}(A^-, y_0) + I_{11}(y_0, A^+) = I_{11}(A^-, y_0) + I_{11}(y_0, A^+) - 2I_{11}(A^-, y_0) = n_1 + \alpha_{11} - \alpha_{11} = n_1$$

$$\int_0^T K_1(\tau)\, u^-(\tau) d\tau = I_{11}(A^-, y_1) - I_{11}(y_1, A^+) = n_1 \tag{9}$$

Let $u(r)$ be an arbitrary function belonging to M. On the set $\sigma(A^-, y_0)$ the function $K_1(r)\ (u^+(r) - u(r))$ is non-positive, and on the set $\sigma(y_0, A^+)$ it is non-negative.

Furthermore,

$$\int_{\sigma(A^-, y_0)} K_1(u^+ - u)\, d\tau = - \int_{\sigma(y_0, A^+)} K_1(u^+ - u)\, d\tau$$

Therefore, applying the mean value theorem, we obtain

$$\int_0^T K_2 u^+ d\tau - \int_0^T K_2 u d\tau = \int_{\sigma(A^-, y_0)} KK_1(u^+ - u)\, d\tau + \int_{\sigma(y_0, A^+)} KK_1(u^+ - u)\, d\tau =$$
$$= K(\tau^*) \int_{\sigma(A^-, y_0)} K_1(u^+ - u)\, d\tau + K(\tau^{**}) \int_{\sigma(y_0, A^+)} K_1(u^+ - u)\, d\tau =$$
$$= (K(\tau^{**}) - K(\tau^*)) \int_{\sigma(y_0, A^+)} K_1(u^+ - u)\, d\tau \geqslant 0$$

since $K(r^{**}) > K(r^*)$.

In a similar manner it can be proved that $u^-(r)$ has the form given above.

Let $n_2(T) \in [n^-, n^+]$. We shall designate by $z = \psi(y)$ a function defined by the relation

$$\varphi(y, z) \equiv I_{11}(A^-, z) - I_{11}(z, y) + I_{11}(y, A^+) = n_1 \tag{10}$$

It follows from (9) that $z = A^-$ if $y = y_0$ and $z = y_1$ if $y = A^+$. The function $\phi(y, z)$ is defined and continuous in the region $A^- < y < A^+$, $A^- < z < y$. For any y in $[y_0, A^+]$ the function $\phi(y, z)$ is monotonically increasing in the sense from the value

$$-I_{11}(A^-, y) + I_{11}(y, A^+) < n_1$$

to the value

$$I_{11}(A^-, y) + I_{11}(y, A^+) = n_1 + \alpha_{11}$$

as z varies from A^- to y.

For any z in $[A^-, A^+]$, the function $\phi(y, z)$ is monotonically decreasing in the strict sense as y varies from z to A^+.

It can be proved therefore that $z = \psi(y)$ is defined, continuous, and monotonically increasing in the strict sense from $z = A^-$ to $z = y_1$ as y varies from y_0 to A^+. It should also be remarked that $\psi(y) < y$ for $y \in [y_0, A^+]$.

In fact, $\psi(y_0) = A^- < y_0$, and if, as y increases, the equality $z^* = \psi(y^*) = y^*$ is satisfied for some y^*, then

$$\varphi(y^*, z^*) = \varphi(y^*, y^*) = n_1 + \alpha_{11}$$

which contradicts the definition $z = \psi(y)$. We now introduce the function

$$\Phi(y) = \int_0^T K_2(\tau)\, u_y(\tau)\, d\tau$$

where

$$u_y(\tau) = m \operatorname{sign} K_1(\tau) \quad \text{if} \quad \tau \in \sigma(A^-, \psi(y)) \cup \sigma(y, A^+)$$

$$u_y(\tau) = -m \operatorname{sign} K_1(\tau) \quad \text{if} \quad \tau \in \sigma(\psi(y), y)$$

it follows from (10) that for any $y \in [y_0, A^+]$ the function $u_y(\tau)$ belongs to the set M.

The function $\Phi(y)$ is defined and continuous in the interval $[y_0, A^+]$ and varies from the value n^- to the value n^+ as y increases from y_0 to A^+; consequently, the equation

$$\Phi(y) = n_2(T) \tag{11}$$

has at least one root $y = a$ in the interval $[y_0, A^+]$, and therefore

$$\int_0^T K_2(\tau)\, u_a(\tau)\, d\tau = n_2(T)$$

that is, $u_a(r)$ coincides with $u_T(r)$.

BIBLIOGRAPHY

1. Fel'dbaum, A.A., Optimal'nye protsessy v sistemakh avtomaticheskogo regulirovaniia (Optimal processes in automatic control systems). *Avtomatika i telemekhanika* Vol. 14, No. 6, 1953.

2. Bellman, R., Glicksberg, I. and Gross, O., On the "bang-bang" control problem. *Quarterly Applied Mathematics* Vol. 14, No. 1, 1956.

3. Krasovskii, N.N., K teorii optimal'nogo regulirovaniia (A contribution to the theory of optimal control). *Avtomatika i telemekhanika* Vol. 18, No. 11, 1957.

4. Voltianskii, V.G., Gamkrelidze, R.V. and Pontriagin, L.S., Teoriia optimal'nykh protsessov. 1. Printsip maksimuma (Theory of Optimal Processes. 1. The Maximum Principle). *Izv. Akad. nauk SSSR* Vol.24, No. 1, 1960.

5. Rozonoer, L.I., Printsip maksimuma v teorii optimal'nykh sistem (The maximum principle in the theory of optimal systems). *Avtomatika i telemekhanika* Vol. 20, No. 11, 1959.

Translated by A.S.